On ψ-Appell polynomials and $Q(\partial_\psi)$-difference calculus nonhomogeneous equation

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Abstract

One discovers why Morgan Ward solution [1] of ψ- *difference calculus* nonhomogeneous equation $\Delta_\psi f = \varphi$ in the form

$$f(x) = \sum_{n\geq 1} \frac{B_n}{n_\psi!}\varphi^{(n-1)}(x) + \int_\psi \varphi(x) + p(x)$$

recently proposed by the present author (see-below) - extends here now to ψ- Appell polynomials case - almost *automatically*. The reason for that is just proper framework i.e. that of the ψ-*Extended Finite Operator Calculus*(**EFOC**) recently being developed and promoted by the present author [2, 3, 4, 5]. Illustrative specifications to q-calculus case and Fibonomial calculus case [5, 6] were already made explicit in [8] due to the of upside down convenient notation for objects of **EFOC** as to be compared with functional formulation [9].

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1 Remark on references usage and the "upside down" notation .

At first let us make a remark on notation.ψ is a number or functions' sequence - sequence of functions of a parameter q. ψ denotes an extension of $\langle\frac{1}{n!}\rangle_{n\geq 0}$ sequence to quite arbitrary one (the so called - "admissible" [2-5]).The specific choices are for example : Fibonomialy-extended sequence $\langle\frac{1}{F_n!}\rangle_{n\geq 0}$ ($\langle F_n\rangle$ - Fibonacci sequence) or just "the usual" ψ-sequence $\langle\frac{1}{n!}\rangle_{n\geq 0}$ or Gauss q-extended $\langle\frac{1}{n_q!}\rangle_{n\geq 0}$ admissible sequence of extended umbral operator calculus, where $n_q = \frac{1-q^n}{1-q}$ and $n_q! = n_q(n-1)_q!, 0_q! = 1$.

The simplicity of calculations is being achieved due to writing objects of these extensions in mnemonic convenient **upside down notation** [2] , [5]

$$\frac{\psi_{(n-1)}}{\psi_n} \equiv n_\psi, n_\psi! = n_\psi(n-1)_\psi!, n > 0, x_\psi \equiv \frac{\psi(x-1)}{\psi(x)}, \tag{1}$$

$$x_\psi^{\underline{k}} = x_\psi(x-1)_\psi(x-2)_\psi...(x-k+1)_\psi \tag{2}$$

$$x_\psi(x-1)_\psi...(x-k+1)_\psi = \frac{\psi(x-1)\psi(x-2)...\psi(x-k)}{\psi(x)\psi(x-1)...\psi(x-k+1)}. \tag{3}$$

If one writes the above in the form $x_\psi \equiv \frac{\psi(x-1)}{\psi(x)} \equiv \Phi(x) \equiv \Phi_x \equiv x_\Phi$, one sees that the name upside down notation is legitimate.

As for references - the papers of main references are: [1, 2, 3]. Consequently we shall then take here notation from [2, 3] and the results from [1] as well as from [2, 3]- for granted. For other references see: [2, 3, 5] (Note the access via ArXiv to [3, 5]).

A_n denotes here ψ- *Appell-Ward numbers* - introduced below.

2 $Q(\partial_\psi)$- difference nonhomogeneous equation

Let us recall [3, 2] the simple fact.

Proposition 2.1. *$Q(\partial_\psi)$ is a ψ- delta operator iff there exists invertible $S \in \Sigma_\psi$ such that $Q(\partial_\psi) = \partial_\psi S$.*

Formally: "$S = Q/\partial_\psi$" or "$S^{-1} = \partial_\psi/Q$". In the sequel we use this abbreviation $Q(\partial_\psi) \equiv Q$.

ψ- *Appell* or generalized *Appell polynomials* $\{A_n(x)\}_{n\geq 0}$ are defined according to

$$\partial_\psi A_n(x) = n_\psi A_{n-1}(x) \tag{4}$$

and they naturally do satisfy the ψ - Sheffer-Appell identity [3, 2]

$$A_n(x +_\psi y) = \sum_{s=0}^{n} \binom{n}{s}_\psi A_s(y) x^{n-s}. \tag{5}$$

ψ-*Appell* or generalized *Appell polynomials* $\{A_n(x)\}_{n\geq 0}$ are equivalently characterized via their ψ- exponential generating function

$$\sum_{n\geq 0} z^n \frac{A_n(x)}{n_\psi!} = A(z) \exp_\psi\{xz\}, \tag{6}$$

where $A(z)$ is a formal series with constant term different from zero - here normalized to one.

The ψ- exponential function of ψ-Appell-Ward numbers $A_n = A_n(0)$ is

$$\sum_{n\geq 0} z^n \frac{A_n}{n_\psi!} = A(z). \tag{7}$$

Naturally ψ- *Appell* $\{A_n(x)\}_{n\geq 0}$ satisfy the ψ- *difference* equation

$$QA_n(x) = n_\psi x^{n-1}; \quad 'n \geq 0, \tag{8}$$

because $QA_n(x) = QS^{-1}x^n = Q(\partial_\psi/Q)x^n = \partial_\psi x^n = n_\psi x^{n-1}$; $n \geq 0$. Therefore they play the same role in $Q(\partial_\psi)$- *difference* calculus as Bernoulli polynomials do in standard difference calculus or ψ-*Bernoulli-Ward polynomials* (see Theorem 16.1 in [1] and consult also [8]) in ψ-*difference* calculus due to the following: The central problem of the $Q(\partial_\psi)$ - *difference calculus* is:

$$Q(\partial_\psi) f = \varphi \qquad \varphi = ?,$$

where f, φ - are for example formal series or polynomials.

The idea of finding solutions is the ψ-*Finite Operator Calculus* [2, 3, 4, 5] standard. As we know (Proposition 2.1, see [2, 3]) any ψ- delta operator Q is of the form $Q(\partial_\psi) = \partial_\psi S$ where$S \in \Sigma_\psi$. Let $Q(\partial_\psi) = \sum_{k\geq 1} \frac{q_k}{k_\psi!}\partial_\psi^k, \ \ q_1 \neq 0$.

Consider then $Q(\partial_\psi = \partial_\psi S)$ with $S = \sum_{k\geq 0} \frac{q_{k+1}}{(k+1)_\psi!}\partial_\psi^k \equiv \sum_{k\geq 0} \frac{s_k}{k_\psi!}\partial_\psi^k;\quad s_0 = q_1 \neq 0$. We have for $S^{-1} \equiv \hat{A}$ - call it: ψ- *Appell operator* - the obvious expression

$$\hat{A} \equiv S^{-1} = \frac{\partial_\psi}{Q_\psi} = \sum_{n\geq 0} \frac{A_n}{n_\psi!}\partial_\psi^n.$$

Now multiply the equation $Q(\partial_\psi f = \varphi)$ by $\hat{A} \equiv \sum_{n\geq 0} \frac{A_n}{n_\psi!}\partial_\psi^n$ thus getting

$$\partial_\psi f = \sum_{n\geq 0} \frac{A_n}{n_\psi!}\varphi^{(n)}, \quad \varphi^{(n)} = \partial_\psi \varphi^{(n-1)}. \tag{9}$$

The solution then reads:

$$f(x) = \sum_{n\geq 1} \frac{A_n}{n_\psi!}\varphi^{(n-1)}(x) + \int_\psi \varphi(x) + p(x), \tag{10}$$

where p is "$Q(\partial_\psi)$- periodic" i.e. $Q(\partial_\psi)p = 0$. Compare with [8] for "$+_\psi 1$-periodic" i.e. $p(x +_\psi 1) = p(x)$ i.e. $\Delta_\psi p = 0$. Here the relevant ψ - integration $\int_\psi \varphi(x)$ is defined as in [2]. We recall it in brief. Let us introduce the following representation for ∂_ψ "difference-ization"

$$\partial_\psi = \hat{n}_\psi \partial_0\ ;\quad \hat{n}_\psi x^{n-1} = n_\psi x^{n-1};\ \ n \geq 1,$$

where $\partial_0 x^n = x^{n-1}$ i.e. ∂_0 is the $q = 0$ Jackson derivative. ∂_0 is identical with divided difference operator. Then we define the linear mapping $\int_\psi$ accordingly:

$$\int_\psi x^n = \left(\hat{x}\frac{1}{\hat{n}_\psi}\right) x^n = \frac{1}{(n+1)_\psi} x^{n+1};\quad n \geq 0$$

where of course $\partial_\psi \circ \int_\psi = id$.

3 Examples

(a) The case of ψ- *Bernoulli-Ward* polynomials and Δ_ψ- *difference calculus* was considered in detail in [8] following [1].

(b) Specification of (a) to the Gauss and Heine originating q-umbral calculus case [1, 2, 3, 4, 5] was already presented in [8].

(c) Specification of (a) to the Lucas originating FFOC - case was also presented in [8] (here: FFOC=**F**ibonomial **F**inite **O**perator **C**alculus), see example 2.1 in [5]). Recall: the *Fibonomial coefficients* (known to Lucas) (F_n- *Fibonacci numbers*) are defined as

$$\binom{n}{k}_F = \frac{F_n!}{F_k!F_{n-k}!} = \binom{n}{n-k}_F,$$

where in up-side down notation: $n_F \equiv F_n \neq 0$,
$n_F! = n_F(n-1)_F(n-2)_F(n-3)_F \ldots 2_F 1_F;\ \ 0_F! = 1;$
$n_F^{\underline{k}} = n_F(n-1)_F \ldots (n-k+1)_F;\quad \binom{n}{k}_F \equiv \frac{n_F^{\underline{k}}}{k_F!}$. We shall call the corresponding linear difference operator $\partial_F;\ \ \partial_F x^n = n_F x^{n-1};\ \ n \geq 0$ the F-derivative. Then in conformity with [1] and with notation as in [2]-[6] one has:

$$E^a(\partial_F) = \sum_{n\geq 0} \frac{a^n}{n_F!}\partial_F^n$$

for the corresponding generalized translation operator $E^a(\partial_F)$. The ψ-integration becomes now still not explored F- integration and we arrive at the$F$- Bernoulli polynomials unknown till now.
Note: recently a combinatorial interpretation of Fibonomial coefficient has been found [6, 7] by the present author.

(d) The other examples of $Q(\partial_\psi)$- *difference calculus* - expected naturally to be of primary importance in applications (for inspiration see [1] and [9]- functional formulation) are provided by the possible use of such ψ-Appell polynomials as:

- ψ-Hermite polynomials $\{H_{n,\psi}\}_{n\geq 0}$:

$$H_{n,\psi}(x) = \left[\sum_{k\geq 0}\left(-\frac{1}{2}\right)^k \frac{\partial_\psi^{2k}}{k_\psi!}\right] x^n \quad n \geq 0;$$

- ψ - Laguerre polynomials $\{L_{n,\psi}\}_{n\geq 0}$ [3]:

$$L_{n,\psi}(x) = \frac{n_q}{n}\hat{x}_\psi \left[\frac{1}{\partial_\psi - 1}\right]^{-n} x^{n-1} = \frac{n_\psi}{n}\hat{x}_\psi \left(\partial_\psi - 1\right)^n x^{n-1} =$$

$$= \frac{n_\psi}{n}\hat{x}_\psi \sum_{k=1}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) \partial_\psi^{n-k} x^{n-1} =$$

$$= \frac{n_\psi}{n} \sum_{k=1}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) (n-1)_{\psi}^{\underline{n-k}} \frac{k}{k_\psi} x^k.$$

For $q = 1$ in q-extended case [9] one recovers the known formula :

$$L_{n,q=1}(x) = \sum_{k=1}^{n} (-1)^k \tfrac{n_q!}{k_q!} \left(\begin{array}{c} n-1 \\ k-1 \end{array} \right) x^k.$$

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